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TRANSLATION

ON THE STABILITY OF THE ROTATING MOVEMENTS OF A SOLID BODY THE CAVITY OF WHICH IS FILLED WITH AN IDEAL LIQUID

By N. G. Chetayev

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ON THE STABILITY OF THE ROTATING MOVEMENTS OF A SOLID BODY THE CAVITY OF WHICH IS FILLED WITH AN IDEAL LIQUID

by
N. G. Chetayev

The problem of the accuracy of the flight of a liquid-filled missile along its trajectory is a hard one if the liquid is viscous. If we consider the liquid in the missile as ideal and incompressible, then we can find the correct solutions of the problems in the stability of rotating movements of this solid body, provided that the interior is completely filled with liquid without any bubbles. In my opinion this solution does not seem unimportant, since starting from it we can consider as sufficient the surplus of stability against the negative influences of viscosity.

1. General Suggestions. Among the many possible displacements of a missile filled with an ideal incompressible liquid free from bubbles, are the following: rotation around any straight line, and the forward displacements like that of a solid body. Hence the movement of this missile is governed by the theorem on moment in a system with its origin in center of gravity of the missile and the liquid O, and with the axes ξ, η, ζ parallel to rigid axes.

These conditions allow us to examine in the problem of the stability of the rotating motions of a missile whose interior is filled with an ideal liquid, only the relative motions, considering the center of gravity O as being fixed.

The axes x, y, z are introduced to facilitate the calculation: let z be the axis of the elliptical inertial rotation of the missile (without liquid) plotted at point O, perhaps not coinciding with the center of gravity of the solid shell; and let the axes x and y lie in the plane perpendicular to the z axis, as may be more convenient for us, but in such a manner that the moments of inertia of the solid shell with reference to the three axes will always be the constant and main axes of the ellipsoid of the inertia of the shell plotted at point O.

It is assumed that the motion of the ideal incompressible liquid filling the cavity of the missile is determined by the instantaneous velocities of the solid shell. Let ρ be the constant density of the liquid ; n, outside normal to the surface S of the cavity occupied by the liquid; and α, β, γ, the directional cosines of the normals on the axes x, y, z. Let p, q, r, stand for instantaneous angular velocities of the rotation of the missile, around the axes x, y, z.

At the beginning of motion the liquid was stationary in the missile. By the Lagrange theory the liquid will move with the potential of the velocities φ (x, y, z, t).

The function φ is determined under the condition that the normal component of the velocity of the liquid coincides with a point on the shell in such a manner that

$$\frac{d\varphi}{dn} = (qz - ry)\alpha + (rx - pz)\beta + (py - qx); \tag{1}$$

Let us assume

$$\varphi = p\psi_1 + q\psi_2 + r\psi_3$$

and let the relationship (1) be satisfied independently from the terms p, q, r. This gives

$$\frac{d\psi_1}{dn} = y\gamma - z\beta, \qquad \frac{d\psi_2}{dn} = z\alpha - x\gamma, \qquad \frac{d\psi_3}{dn} = x\beta - y\alpha \tag{2}$$

Hence ψ, does not depend on t, but depends on the cavity which is filled with an incompressible ideal liquid.

The kinetic energy of the rotating motion of the missile's shell is

$$2T' = A'p^2 + B'q^2 + C'r^2$$

where A', B', C' are the constant moments of inertia of the missile's solid shell together with the dividing walls of the cavity with respect to axes x, y, z. Kinetic energy of relative motions of the liquid, in the cavity

$$2T^{*}=\rho\iiint\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right]d\tau=\sum\omega_i\omega_j\{\psi_i,\ \psi_j\}$$

$$\omega_1=p,\qquad \omega_2=q,\qquad \omega_3=r$$

$$\{\psi_i,\ \psi_j\}=\rho\iiint\left[\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}+\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}+\frac{\partial\psi_i}{\partial z}\frac{\partial\psi_j}{\partial z}\right]d\tau$$

2. <u>Cavity in the Form of a Circular Cylinder</u>. Let us examine the cavity in the form of a circular cylinder with radius a, axis z, and height 2h. Since, in respect to any straight line, passing through the point 0 and perpendicular to the axis z, the moment of inertia of the hard shell has only one value, it is easier to determine the position of the missile by using the coordinate system $\xi\eta\zeta$, connected with the missile, (Figure 1). (see page 17)

The system of rectangular axes $(\xi\eta\zeta)$ when rotated around the axis n at an angle α will change into the system (x,η,j), but the system (x,η,j) when rotated around the axis x at an angle β changes into the system (xyz). The angle of rotation of the missile around its axis z in the system (xyz) we will denote by ω.

From the diagram we must conclude that the angles, α, β, ω, are holonomous coordinates of the missile, and hence the differential equation of the rotating motions of the missile with these variables will resemble Lagrange equation.

From the diagram we conclude that the projection of the absolute instantaneous velocity of the rotating solid shell along the axes x, y, z are

$$p=\beta',\qquad q=\alpha'\cos\beta,\qquad r=-\alpha'\sin\beta+\omega'$$

Where α', β', ω' are derivatives of time t corresponding to the angles α, β, ω.

Total kinetic energy of the missile together with its liquid in the interior will be $T = T' + T^*$.

Similarly to the classic example of the stability of the rotating motions of a conventional shell with solid equipment on a flat trajectory, let us consider only the resisting couple with moment $\mu\sin\gamma$, , proportional to the sine of the angle γ between the z axis of the missile and the velocity of the center of gravity in the system 0.

Assuming that the axis ζ is directed along velocity of the center of gravity in system O, then we shall have $\cos\gamma = \cos\alpha\cos\beta$.

The possible work of the resisting couple is simply

$$\mu\sin\gamma\delta\gamma = \delta(-\mu\cos\gamma) = \delta(-\mu\cos\alpha\cos\beta) =$$
$$= \mu\sin\alpha\cos\beta\delta\alpha + \mu\cos\alpha\sin\beta\delta\beta = Q_\alpha\delta\alpha + Q_\beta\delta\beta + Q_\omega\delta\omega$$

From this the generalized forces $Q_\alpha, Q_\beta, Q_\omega$ are determined by the formulas

$$Q_\alpha = \mu\sin\alpha\cos\beta, \qquad Q_\beta = \mu\cos\alpha\sin\beta, \qquad Q_\omega = 0$$

In order to set up the differential equations of rotation motions of the missile filled with liquid, one must calculate T*. The problem of determining the function ψ was solved by N. E. Zhukobskiy. The solution is given below with insignificant amplification.

The boundary condition of function ψ_3 on the surface S of the hollow gives $d\psi_3/dn = 0$, since on the lateral surface of the cylinder $\alpha/x = \beta/y$ and on the bases of the cylinder $\alpha = 0,\ \beta = 0$. Here by Neumann's principle we have $\psi_3 =$ const, and consequently

$$[\psi_1, \psi_3] = 0, \qquad [\psi_2, \psi_3] = 0, \qquad [\psi_3, \psi_3] = 0$$

From the axial symmetry of the cavity it can be concluded, that it is sufficient to determine only one of the functions ψ_1, ψ_2. Let us determine ψ_1. Let us assume

$$\psi_1 = F_1 - yz$$

The boundary condition (2) for function ψ_1, gives F_1, as condition to S

$$\frac{dF_1}{dn} = 2y\gamma$$

And hence the problem consists of determining the harmonic function F_1 ($\Delta F_1 = 0$), which on the lateral surface of the cylinder S satisfies the condition

$$\frac{dF_1}{dn} = 0 \qquad (3)$$

and on the bases of the cylinder

$$\frac{dF_1}{dn} = \mp 2y \qquad (4)$$

where the minus and the plus signs are taken from the lower (U), and the upper (O) bases.

In the cylindrioal coordinates the Laplace equation takes on the following form

$$\frac{\partial^2 F_1}{\partial r^2} + \frac{1}{r}\frac{\partial F_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_1}{\partial \theta^2} + \frac{\partial^2 F_1}{\partial z^2} = 0 \tag{5}$$

Let us assume

$$F_1 = \sin\theta \sum_n C_n R_n \operatorname{sh}\left\{\lambda_n\left(z - \frac{z_0 - z_u}{2}\right)\right\}$$

where C_n are oonstants, R_n is the function of r only, Z_o is the ooordinate of the oylinder upper base S, Z_u is the ooordinate of the lower base. From the Laplace equations for F_1, it follows that the functions R_n must satisfy the equation

$$\frac{d^2 R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} + \left(\lambda_n^2 - \frac{1}{r^2}\right)R_n = 0$$

By introduoing new variables $\zeta = r\lambda_n$, we equate this equation to Bessel's equation

$$\frac{d^2 R}{d\zeta^2} + \frac{1}{\zeta}\frac{dR}{d\zeta} + \left(1 - \frac{1}{\zeta^2}\right)R = 0$$

By the sense of the problem under consideration with r = 0, one should get R = 0, hence along the z axis of the cylindrical cavity S, the liquid must not be of infinite values for the velooity, but limited with r = 0, the magnitude should be $r^{-1}\partial F_1/\partial\theta$.

Therefore, the following should be accepted

$$R = J_1(\zeta) = \frac{\zeta}{2\pi}\int_0^\pi \cos(\zeta\cos\theta)\sin^2\theta\, d\theta$$

where $J_1(\zeta)$ is Bessel's integral of the first order and first degree. The boundary oondition (3) on the lateral surface S gives the relationship $dF_1/dr = 0$, or $dR_n/dr = 0$, , or better

$$\frac{dJ_1(\zeta)}{d\zeta} = 0$$

N. E. Zhukovskiy determined the first roots of this equation

$$\zeta_1 = 1.8412, \quad \zeta_2 = 5.4315, \quad \zeta_3 = 8.5363$$
$$\zeta_4 = 11.7060, \quad \zeta_5 = 14.8633, \quad \zeta_6 = 18.0155$$

The values of the constants λ_n, are determined by the formula $\lambda_n = \frac{\zeta_n}{a}$

The boundary conditions (4) for the lower and upper bases gives. the following relationship

$$\sum C_n \lambda_n J_1(\lambda_n r) \operatorname{ch} \lambda_n h = 2r$$

Since $Z_o - Z_u = 2h$. Hence, by using the well known formulas,

$$\int_0^a J_1(\lambda_n r) J_1(\lambda_m r)\, r\, dr = \begin{cases} 0, & \text{если } n \neq m \\ \frac{\lambda_n^2 a^2 - 1}{2\lambda_n^2} [J_1(\lambda_n a)]^2, & \text{если } n = m \end{cases}$$

the constants C_n are determined:

$$C_n = \frac{4\lambda_n}{(\lambda_n^2 a^2 - 1)[J_1(\lambda_n a)]^2 \operatorname{ch}(\lambda_n h)} \int_0^a J_1(\lambda_n r) r^2 dr$$

but

$$\int_0^a J_1(\lambda_n r) r^2 dr = \frac{J_1(\zeta_n)\zeta_n}{\lambda_n^3} \qquad (\zeta_n = \lambda_n a)$$

Hence

$$F_1 = 4a^2 \sin\theta \sum_n \frac{1}{\zeta(\zeta_n^2 - 1)} \frac{J_1(\zeta_n r/a)}{J_1(\zeta_n)} \frac{1}{\operatorname{ch}(\zeta_n h/a)} \operatorname{sh}\left(\zeta_n \frac{2z - z_0 - z_n}{a}\right)$$

Let us compute $[\psi_1, \psi_1]$. If we take into account the boundary conditions (3) and (4), we will have

$$[\psi_1, \psi_1] = \rho \iiint \left[\left(\frac{\partial F_1}{\partial x}\right)^2 + \left(\frac{\partial F_1}{\partial y} - z\right)^2 + \left(\frac{\partial F_1}{\partial z} - y\right)^2\right] d\tau =$$
$$= \rho \iiint (y^2 + z^2)\, d\tau + \rho \iint_C (F_1^{(0)} - F_1^{(n)})\, 2y\, d\sigma - 4\rho (z_0 - z_n) \iint_C y^2 d\sigma$$

where C is the base of cylindrical cavity. By substituting the calculated value of the function F_1 we obtain

$$[\psi_1, \psi_1] = M \frac{z_0^2 + z_0 z_n + z_n^2}{3} - Ma^2 \left[\frac{3}{4} - 8\frac{a}{h} \sum_n \frac{1}{\zeta_n^3(\zeta_n^2 - 1)} \operatorname{th}\left(\zeta_n \frac{h}{a}\right)\right] \qquad (6)$$

where $M = 2\pi\rho a^2 h$ is the mass of the liquid, filling the inside of the shell. This formula coincides with the formula of N. E. Zhurovskiy with $Z_o = -Z_u = h$.

From the symmetry of the cavity, with respect to the z axis, there follows the equality $[\psi_1, \psi_1] = [\psi_2, \psi_2]$; . It is also difficult to determine that $[\psi_1, \psi_2] = 0$. . Let us designate

$$A^* = [\psi_1, \psi_1]$$

From this the kinetic energy of the rotating motions of the missile that is filled with liquid will be

$$2T = (A' + A^*)(p^2 + q^2) + C'r^2$$

The stability problem of rotating motions of such a missile is parallel to the classic problem of the stability of a simple missile with the moment of inertia of A = B = A' + A*, C = C'. Hence (3) one can immediately write the stability condition of rotating motions of the liquid filled missiles on flat trajectories:

$$C'^2 r^2 - 4(A' + A^*)\mu > 0$$

For an ideal incompressible liquid, and for the same type of missile as studied above (C', A' = constants) and for the same conditions of launching (r, μ = constants) the density of liquid, in increasing the term A*, has a negative effect upon the conditions of stability.

3. Cylindrical Cavity with One Flat Diaphram. In order to increase the axial moment of inertia of a missile filled with liquid and thus increase the stability of rotating motions of the missile on its trajectory, the diaphrams are placed in the cylindrical cavity.

Let us study the influence of one flat diaphram set radially in the cavity.

A missile with one diaphram does not possess axial symmetry. Hence, to simplify the calculations for the x, y, z axes, we will choose axes tied with rigid shell, with the origin in the center of gravity O of the filled missile so that the plane y = 0 would be the plane of the diaphram.

Projections of instantaneous absolute angular velocity of the missile on this axis will be

$$p = \beta' \cos\omega + \alpha' \sin\omega \cos\beta$$
$$q = -\beta' \sin\omega + \alpha' \cos\omega \cos\beta$$
$$r = \omega' - \alpha' \sin\beta$$

Kinetic energy of rotating motions of the rigid shell will be

$$2T' = A'p^2 + B'q^2 + C'r^2$$

where the constants A', B', C' represents the moments of inertia of the missile with respect to the axes x, y, z. The kinetic energy of the liquid will be expressed by the general formula

$$2T^* = \sum \omega_i \omega_j [\psi_i, \psi_j]$$

It is necessary to determine the function ψ_i. The function ψ_3 was found by Stock.

We will study that part of the cavity which is divided by the diaphram and in which ($(x = r\cos\theta,\ y = r\sin\theta)$) the angle θ varies from 0 to π. Let us assume

$$\psi_3 = xy + F_3$$

The condition (2) on the boundary of the cavity gives the relationship

$$dF_3/dn = -2ya$$

From this, on the wall of the cylinder ($(0 < \theta < \pi)$) we have

$$\partial F_3/\partial r = -a \sin 2\theta \qquad (r = a) \tag{7}$$

on the diaphram $(\theta = 0,\ \theta = \pi)$

$$\partial F_3/\partial r\theta = 0 \tag{8}$$

at the sections of the lower and upper bases

$$\partial F_3/\partial z = 0 \tag{9}$$

We will try to find function F_3 in the form

$$F_3 = \sum_n^{\infty} C_n r^n \cos n\theta$$

The boundary condition (9) is satisfied immediately, sinoe F does not depend on Z condition (8) is also fulfilled, since the walls of the diaphram y = 0 we have $\theta = 0$, $\theta = \pi$. If we use the formula

$$\sin 2\theta = -\frac{8}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\theta}{(2k-1)^2-4} \qquad (0 < \theta < \pi)$$

then the boundary oondition (7) oan be written in the form

$$\sum_{n} C_n n a^{n-1}\cos n\theta = \frac{8a}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\theta}{(2k-1)^2-4}$$

From this

$$C_{2k} = 0, \qquad C_{2k-1} = \frac{8a^{3-2k}}{\pi(2k-1)[(2k-1)^2-4]}$$

For the half ($y > 0$) of the circular oavity under oonsideration we oan oaloulate

$$[\psi_3, \psi_3] = Ma^2\left(\frac{5}{4} - \frac{32}{\pi^3}\sum_{n}\frac{1}{n(n^2-4)^2}\right) \qquad (n = 1, 3, 5, \ldots) \tag{10}$$

where as before $M = 2\pi\rho a^2 h$. We can show that

$$[\psi_1, \psi_3] = 0, \qquad [\psi_2, \psi_3] = 0$$

Let us oaloulate the funotion ψ_1. Let $\psi_1 = F_1 - yz$; the boundary conditions (2) give us the following relationship on the boundary of the oavity under consideration

$$dF_1/dn = 2y\gamma$$

In other words, on the lateral surfaoe, sinoe $\gamma = 0$, it should be

$$dF_1/dn = 0 \tag{11}$$

and on the bases

$$dF_1/dn = \mp 2y \tag{12}$$

where the plus and minus signs are taken for the lower and upper bases, respectively.

Inside the oavity the funotion F_1 must be harmonic $\Delta F_1 = 0$; in cylindrical coordinates it must satisfy equation (5)

Let us assume

$$F_1 = \sum_{n,m} C_{nm} \cos n\theta R_{nm} \operatorname{sh} \lambda_{nm} \left(z - \frac{z_0 + z_u}{2} \right).$$

where R_{mm} denotes only the function of r. Function R_{nm} must satisfy the equation

$$R''_{nm} + \frac{R'_{nm}}{r} + \left(\lambda_{nm} - \frac{n^2}{r^2} \right) R_{nm} = 0$$

If we introduce a new variable $\zeta = \lambda_{nm} r$, then it will be the familiar Bessel equation

$$\frac{d^2 R_{nm}}{d\zeta^2} + \frac{1}{\zeta} \frac{dR_{nm}}{d\zeta} + \left(1 - \frac{n^2}{\zeta^2} \right) R_{nm} = 0$$

For this problem we must take

$$R_{nm} = J_n(\zeta) = \frac{1}{\pi} \int_0^{\pi} \cos(\zeta \sin z - nz)\, dz$$

where $J_n(\zeta)$ denotes the Bessel integral of the nth order and first degree.

On the surface of the cylinder ($(r = a,\ 0 < \theta < \pi)$) from boundary condition (11) we derive the equations

$$dJ_n(\zeta)/d\zeta = 0 \qquad (\zeta = \lambda_{nm} a)$$

determining the eigen values of λ_{nm}. The boundary condition (11) on the diaphram ($y = 0$, $\theta = 0$, $\theta = \pi$) is immediately satisfied. In accordance with the condition (12) for the upper and lower bases we have ($Z_o - Z_u = 2h$)

$$\sum_{n,m} C_{nm} \lambda_{nm} \cos n\theta R_{nm} \operatorname{ch}(\lambda_{nm} h) = 2r \sin\theta \tag{13}$$

For the value θ within the interval ($0, \pi$) the function sin θ may be expanded into Fourier series in accordance with the cosines.

$$\sin\theta = -\frac{4}{\pi} \sum \frac{\cos n\theta}{n^2 - 1} \qquad (n = 0, 2, 4, \ldots)$$

By substituting this series in formula (13) and equating the coefficients of Cos θ , we will obtain for n = 0, 2, 4, . . .

$$\sum_m C_{nm} \lambda_{nm} J_n(\lambda_{nm} r) \operatorname{ch}(\lambda_{nm} h) = -\frac{8}{\pi} \frac{r}{n^2 - 1} \tag{14}$$

and in the first part (of equation) zeros with n = 1, 3, 5, ...

$$\int_0^a rJ_n(\lambda_{nk}r)\,J_n(\lambda_{nm}r)\,dr = \begin{cases} 0 & \text{при } \lambda_{nk} \neq \lambda_{nm} \\ \dfrac{\lambda_{nm}^2 a - 1}{2\lambda_{nm}^2}[J_n(\lambda_{nm}a)]^2 & \text{при } \lambda_{nm} = \lambda_{nk} \end{cases}$$

the following terms of the constants(are determined) C_{nm} (n = 0, 2, 4, ...)

$$C_{nm} = -\frac{16}{\pi(n^2-1)}\,\frac{\lambda_{nm}}{(\lambda_{nm}^2 a^2 - 1)\operatorname{ch}(\lambda_{nm}h)\,[J_n(\lambda_{nm}a)]^2}\int_0^a r^2 J_n(\lambda_{nm}r)\,dr$$

For odd terms n, the constants C_{nm} are zeros.

Substituting these constant terms in the function F_1 and the latter in the already used expression (2) $[\psi_1, \psi_1]$, after calculation we obtain

$$[\psi_1, \psi_1] = \frac{M}{2}\left[\frac{z_0^2 + z_0 z_n + z_n^2}{3} - \frac{3}{4}a^2 + \right.$$

$$\left. + \frac{128}{\pi^2 a^2 h}\sum_{n,m}\frac{\lambda_{nm}\operatorname{th}(\lambda_{nm}h)}{(\lambda_{nm}^2 a^2 - 1)(n^2-1)^2}\left(\frac{1}{J_n(\lambda_{nm}a)}\int_0^a r^2 J_n(\lambda_{nm}r)\,dr\right)^2\right]$$

Now we will find function ψ_2. With this in mind let

$$\psi_2 = xz + F_2$$

The boundary condition on the surface of the examined cavity will be

$$dF_2/dn = -2x\gamma$$

This means, that on the lateral surface of the cavity $(\gamma = 0)$ we must have

$$dF_2/dn = 0$$

and on the bases

$$dF_2/dn = \pm 2x$$

where the plus and minus signs are taken for the lower and upper bases respectively.

We will seek function F_2 in the form of the series

$$F_2 = \sum_n C_n \cos\theta J_1(\lambda_n r)\operatorname{sh}\lambda_n\left(z - \frac{z_0 + z_n}{2}\right)$$

It is on that part of the surface of the cylinder which enters the boundaries of the half of the cavity under consideration $0 \leqslant \theta \leqslant \pi$, that by virtue of boundary conditions we have the equation

$$\frac{dJ_1(\lambda_n a)}{da} = 0$$

determining the constant terms λ_n.

On the flat diaphram the boundary condition is satisfied, since there $\sin\theta = 0$.

From boundary condition of the bases of the cavity we get

$$\sum_n C_n J_1(\lambda_n r)\lambda_n \operatorname{ch}(\lambda_n h) = -2r$$

Hence

$$C_n = \frac{4a^2}{\zeta_n(\zeta_n^2-1)\operatorname{ch}(\lambda_n h) J_1(\zeta)} \qquad (\zeta_n = \lambda_n a)$$

Therefore

$$F_2 = -4a^2\cos\theta \sum_n \frac{1}{\zeta_n(\zeta_n^2-1)} \frac{J_1(\zeta_n r/a)}{J_1(\zeta_n)} \frac{1}{\operatorname{ch}(\zeta_n h/a)} \operatorname{sh}\left(\zeta_n \frac{2z - z_0 - z_n}{2}\right)$$

Substituting this expression for F_2 in (ψ_2, ψ_2) after calculations we get

$$[\psi_2, \psi_2] = \frac{M}{2}\left[\frac{z_0^2 + z_0 z_n + z_n^2}{3} + \frac{5}{4}a^2 - 4a^2\frac{a}{h}\sum_n \frac{1}{\zeta_n^3(\zeta_n^2-1)} \operatorname{th}\left(\zeta_n \frac{h}{a}\right)\right]$$

For the cavity under discussion $(\psi_1, \psi_2) = 0$.

The moments of inertia of the accompanying mass of the liquid filling the cylindrical cavity, which is divided by one flat diaphram, will be

$$A^* = 2[\psi_1, \psi_1], \qquad B^* = 2[\psi_2, \psi_2], \qquad C^* = 2[\psi_3, \psi_3]$$

The kinetic energy of the rotating motions of the missile together with the liquid filling it, will be

$$2T = A_{p2} = B_{q}2 = C_{r}2$$

where

$$A = A' + A^*, \qquad B = V' + B^*, \qquad C = C' + C^*$$

From this we know that the ellipsoid of the inertia of the missile and accompanying masses will be triaxial in this case. Such a case of an elongated missile with solid filling has not been completely investigated for its stability. It is necessary to examine it separately.

With this in mind, and using for the Lagrange variables the angles α, β, ω, we set up equations of the rotating motions of the missile in the form of the Lagrange equations. In doing so, we take as the generalized forces, as was explained in (1)

$$Q_\alpha = \mu \sin\alpha \cos\beta, \qquad Q_\beta = \mu \cos\alpha \sin\beta, \qquad Q_\omega = 0$$

The obvious form of Lagrange equation in this case will be

$$C\frac{d}{dt}(\omega' - \alpha' \sin\beta) = (A - B)(-\beta'^2 + \alpha'^2 \cos^2\beta)\cos\omega \sin\omega + $$
$$+ (A - B)\alpha'\beta'(\cos^2\omega - \sin^2\omega)\cos\beta$$

$$\frac{d}{dt}[A(\beta'\cos\omega + \alpha'\sin\omega\cos\beta)\sin\omega\cos\beta +$$
$$+ B(-\beta'\sin\omega + \alpha'\cos\omega\cos\beta)\cos\omega\cos\beta -$$
$$- C(\omega' - \alpha'\sin\beta)\sin\beta] = \mu\sin\alpha\cos\beta$$

$$\frac{d}{dt}[A(\beta'\cos\omega + \alpha'\sin\omega\cos\beta)\cos\omega - B(-\beta'\sin\omega + \alpha'\cos\omega\cos\beta)\sin\omega] =$$
$$= -[A(\beta'\cos\omega + \alpha'\sin\omega\cos\beta)\alpha'\sin\omega\sin\beta +$$
$$+ B(-\beta'\sin\omega + \alpha'\cos\omega\cos\beta)\alpha'\cos\omega\sin\beta +$$
$$+ C(\omega' - \alpha'\sin\beta)\alpha'\cos\beta] + \mu\cos\alpha\sin\beta$$

These equations have a particular solution

$$\omega' = \omega_0', \qquad \alpha = 0, \qquad \beta = 0$$

Let us take this particular solution for the undisturbed motion and try to investigate its stability in the first approximation. Let $\omega_0 = \omega_0' t$, $\omega - \omega_0 = \xi$. The equations in the variations for this undisturbed motion are of the form

$$C\xi'' = 0$$

$$A(\beta''\cos\omega_0 + \alpha'\sin\omega_0) + (A - B)\omega_0'(-\beta'\sin\omega_0 + \alpha\cos\omega_0) =$$
$$= -C\omega_0'(-\beta'\sin\omega_0 + \alpha'\cos\omega_0) + \mu(\beta\cos\omega_0 + \alpha\sin\omega_0)$$

$$B(-\beta''\sin\omega_0 + \alpha''\cos\omega_0) + (A - B)\omega_0'(\beta'\cos\omega_0 + \alpha'\sin\omega_0) =$$
$$= C\omega_0'(\beta'\cos\omega_0 + \alpha'\sin\omega_0) + \mu(-\beta\sin\omega_0 + \alpha\cos\omega_0)$$

This system presents equations with the periodic coefficients ($\omega_0 = \omega_0' t$).

As Lyapunov established, such systems of differential equations can be transformed into a system with constant coefficients, without changing the problem of stability. The first of the equations is of no particular interest; it expresses the obvious properties of the angle ω when the given terms are affected in the first approximation. In order to transform the last two equations, we will introduce new variables

$$u = \alpha \sin \omega_0 + \beta \cos \omega_0, \qquad v = \alpha \cos \omega_0 - \beta \sin \omega_0$$

The determinant of the transformation is different from zero and = − 1. Hence

$$\begin{aligned}
\alpha' \sin \omega_0 + \beta' \cos \omega_0 &= u' - \omega_0' v \\
\alpha' \cos \omega_0 - \beta' \sin \omega_0 &= v' + \omega_0' u \\
\alpha'' \sin \omega_0 + \beta'' \cos \omega_0 &= u'' - 2\omega_0' v' - \omega_0'^2 u \\
\alpha'' \cos \omega_0 - \beta'' \sin \omega_0 &= v'' + 2\omega_0' u' - \omega_0'^2 v
\end{aligned}$$

It follows then that the equations in the variations with new variables will be

$$\begin{aligned}
A(u'' - 2\omega_0' v' - \omega_0'^2 u) &= \mu u + C\omega_0'(-v' - \omega_0' u) \\
B(v'' + 2\omega_0' u' - \omega_0'^2 v) &= \mu v + C\omega_0'(u' - \omega_0' v)
\end{aligned}$$

or better

$$\begin{aligned}
Au'' + [(C - B)\omega_0'^2 - \mu] u + (C - A - B)\omega_0' v' &= 0 \\
Bv'' + [(C - A)\omega_0'^2 - \mu] v - (C - A - B)\omega_0' u' &= 0
\end{aligned}$$

The characteristic equation of this system of differential equations with constant coefficients is

$$\begin{aligned}
AB\lambda^4 + \{[AB + (A - C)(B - C)]\omega_0'^2 - \mu(A - B)\}\lambda^2 + \\
+ [(C - B)\omega_0'^2 - \mu][(C - A)\omega_0'^2 - \mu] = 0
\end{aligned}$$

The conditions, that the roots of this equation are purely imaginary, are expressed by the following three inequalities

$$\begin{aligned}
[(C - A)\omega_0'^2 - \mu][(C - B)\omega_0'^2 - \mu] > 0 \\
[AB + (A - C)(B - C)]\omega_0'^2 - \mu(A + B) > 0 \\
[(AB + (A - C)(B - C))\omega_0'^2 - \mu(A + B)]^2 - \\
- 4AB[(C - A)\omega_0'^2 - \mu][(C - B)\omega_0'^2 - \mu] > 0
\end{aligned}$$

We note, in the last condition when A = B, leads us to the well-known Mayevskiy theorem of the stability of the rotating motions of a missile; in this case the first two inequalities will be satisfied.

4. Cylindrical Cavity with Two Crossed Diaphrams. Let us examine the case where two diaphrams are orthogonally placed with respect to each other in the cylindrical cavity. Since the ellipsoid of inertia of the missile's rigid shell will be the ellipsoid of rotation around the axis Z, the axes of the coordinates bound to the diaphrams are so chosen, that their equations will be x = 0, y = 0 and later in the dynamic part of this paragraph we will change to the axes used in (2).

Function ψ_3 was determined by Stokes taking the limit in calculating the value of one unknown. There is no point in changing Stokes' calculations, and we will accept the expression (ψ_3, ψ_3) as derived by Zhukovskiy. For the cavity under discussion

$$[\psi_1, \psi_3]=0, \qquad [\psi_2, \psi_3]=0, \qquad [\psi_1, \psi_2]=0$$

and it is evident that after finding function ψ_1 the problem of finding function ψ_2 does not arise, since

$$[\psi_1, \psi_1]=[\psi_2, \psi_2]$$

For the purpose of computing ψ_1, let $\psi_1=-yz+F_1$

Boundary conditions (2) give

$$\frac{dF_1}{dn}=2y\gamma$$

On the lateral surface of the cavity $0\leqslant\theta\leqslant 1/2\pi$, separated by the crossed diaphrams $\gamma=0$ and it should be $$\frac{dF_1}{dn}=0$$

and on the bases $$\frac{dF_1}{dn}=\mp 2y$$

where the minus and plus signs are taken for the lower and upper bases.

We shall look for the harmonic function F_1, in the form of series expansion (n = 0, 2, 4,...)

$$F_1 = \sum_{n,m} C_{nm} \cos n\theta J_n(\lambda_{nm} r) \operatorname{sh}\left(\lambda_{nm}\left(z - \frac{z_0 + z_u}{2}\right)\right)$$

On the surface of the cylinder limiting the cavity $0 \leqslant \theta \leqslant {}^1/_2\pi$, under discussion, we have in accordance with boundary conditions

$$\frac{dJ_n(\lambda_{nm} a)}{da} = 0$$

These equations determine the value λ_{nm}. The boundary conditions on the diaphram will be satisfied, since when n is even on the diaphram we will have $\sin n\theta = 0$. The boundary conditions on the upper and lower boundaries bases give

$$\sum_{n,m} C_{nm} \lambda_{nm} \cos n\theta J_n(\lambda_{nm} r) \operatorname{ch}(\lambda_{nm} h) = 2r \sin\theta$$

By taking $0 \leqslant \theta \leqslant {}^1/_2\pi$ used in series expansion of $\sin\theta$ in the preceding paragraph we obtain the established values of Cmm when (n = 0, 2, 4,...).

$$[\psi_1, \psi_1] = \frac{M}{4}\left(\frac{z_0^2 + z_0 z_u + z_u^2}{3} - \frac{3a^2}{4}\right) +$$
$$+ \frac{64\rho}{4} \sum_{n,m} \frac{1}{(n-1)^2} \frac{\lambda_{nm} \operatorname{th}(\lambda_{nm} h)}{\lambda_{nm}^2 a^2 - 1} \left(\frac{1}{J_n(\lambda_{nm} a)} \int_0^a r^2 J_n(\lambda_{nm} r) dr\right)$$

From this the moments of inertia of the accompanying masses of the liquid that fill up the four sections of the cavity separated by the orthogonal diaphrams will be

$$A^* = B^* = [\psi_1, \psi_1], \qquad C^* = 4\,[\,[\psi_3, \psi_3]\,].$$

The ellipsoid of the inertia of a missile that is filled with an ideal liquid will be the ellipse of the missile's rotation along its axis of propagation. The problem of the stability of rotating motions of such a missile coincides with the classical problem, and hence we can immediately write the condition for stability

$$(C' + C^*)^2 r^2 - 4(A' + A^*)\mu > 0$$

On the left side is the second degree polynomial of the liquid's density . Hence the stability of the cavity will be determined by the disposition of the real roots of this polynomial.

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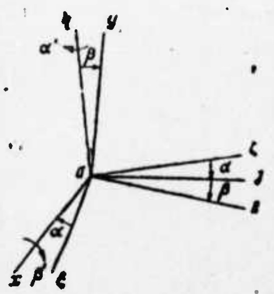

Figure 1.